FARADAY'S LAW AND INDUCTION

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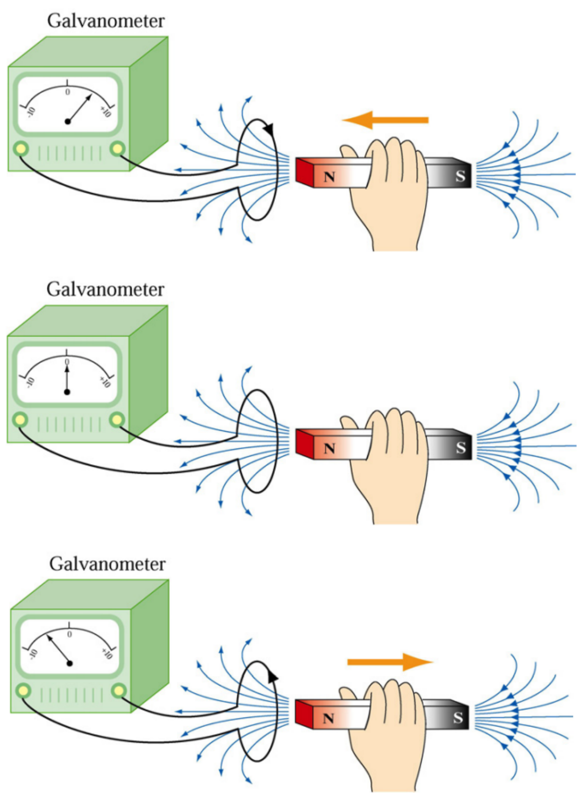
1. WHAT IS ELECTROMAGNETIC INDUCTION?

Electromagnetic induction is the process by which a current can be *induced* to flow due to a changing magnetic field. An electric current produces a magnetic field – and conversely, a changing magnetic field generates electric current.



2. FARADAY'S LAW OF INDUCTION

Theorem 1 (Faraday's Law of Induction)**.** *The induced emf ε is proportional to the negative rate of change of magnetic flux:*

$$\boxed{\varepsilon = -\frac{d\Phi_B}{dt},}$$

where $\Phi_B = \int \vec{B} \cdot d\vec{A} = BA\cos\theta$.

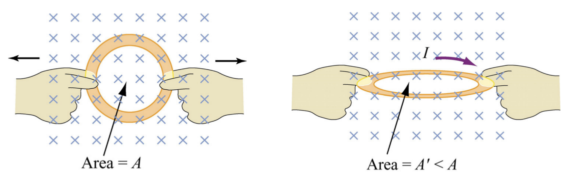
For a coil with N loops, the induced emf is N times as large:

$$\varepsilon = -N\frac{d\Phi_B}{dt}.$$

Thus, emf can be induced in three ways:

(1) By varying the magnitude of $\vec{B}$ over time;

(2) By varying the magnitude of $\vec{A}$;



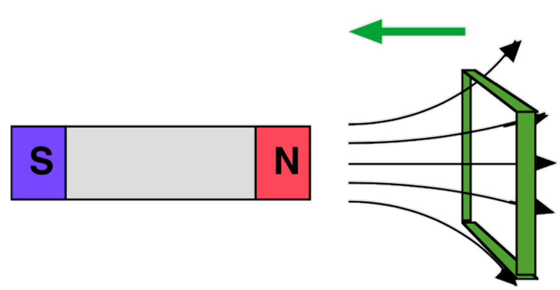
(3) By varying the angle between $\vec{B}$ and $\vec{A}$ with time.

3. Lenz's Law

Faraday's Law allows us to calculate the magnitude of induced current, but in order to determine the direction of current flow, we use Lenz's Law.

Theorem 2 (Lenz's Law)**.** *The induced current produces a magnetic field that opposes the change in magnetic flux.*

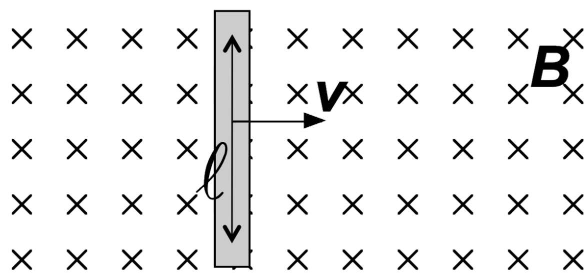
Example.



In the picture above, with a stationary magnet and a conducting loop moving to the left, the direction of current induced is up in front, down in back. The magnetic flux is increasing, so the induced current has a magnetic field in the opposite direction as B. Use the right hand rule (aligning your thumb with the direction of the magnetic field and curling your fingers).

4. Motional EMF

Just as there is emf produced by a changing magnetic flux, there is emf produced by moving a conductor through a magnetic field.



As free charges in the conductor are moved through the magnetic field with velocity v, they experience a magnetic force $\vec{F_B} = q\vec{v} \times \vec{B}$. Positive charges are forced upward, creating an electric field downward due to the separation of charges.

Eventually, the magnetic force upward and the electric force downward will reach equilibrium, when $qvB = qE$, or $E = vB$.

At equilibrium we can find the electric potential difference:

$$V = Ed = \boxed{Blv.}$$

Oftentimes motional questions include:

- Identify emf or current (Faraday's Law)
- Determine direction of current (Lenz's Law)
- Calculate magnetic force on the bar/conductor ($\vec{F_B} = I\vec{L} \times \vec{B}$, then $I = \frac{V}{R}$)
- Power to move bar = power dissipated by circuit ($P = Fv = \frac{V^2}{R}$)

5. Eddy Currents

If a solid conductor (rather than a loop) moves through a magnetic field, current is still induced as a consequence of changing magnetic flux. The free charges in the conductor circulate and produce *eddy current*.

6. Self-Inductance

In a simple circuit with a battery, resistor, and switch, throwing the switch does not immediately initiate constant current flow. Instead, as current increases, the current-carrying wire produces an magnetic field. The induced current fights the increasing magnetic flux so that the net current increases gradually.

Definition 3 (Inductance)**.** Inductance, or L, represents the resistance to change in current in a circuit.

Inductance varies inversely with changing current:

Theorem 4.

$$\varepsilon = -L\frac{dI}{dt}$$

7. Inductance of a Coil and Solenoid

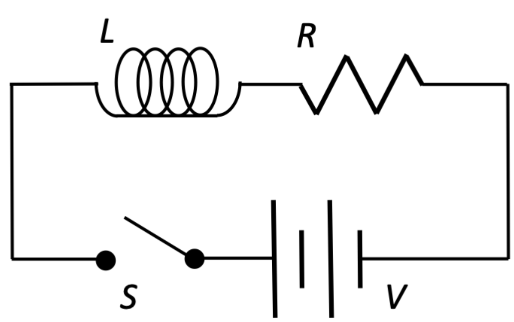
Theorem 5 (Inductance of Coil)**.**

$$L_{coil} = N\frac{\Phi_B}{I}$$

Theorem 6 (Inductance of Solenoid)**.**

$$L_{solenoid} = \mu_0 n^2 V$$

8. RL Circuit



Using Kirchoff's Loop Rule:

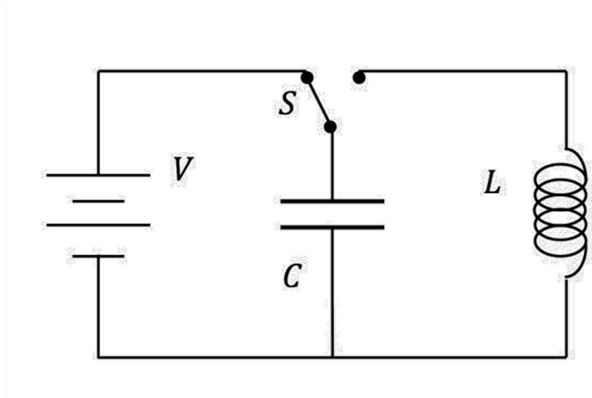
$$I = \frac{V}{R}(1 - e^{-\frac{t}{\tau}}),$$

where $\tau = \frac{L}{R}$.

Theorem 7 (Energy in Inductor)**.**

$$U = \frac{1}{2}LI^2.$$

9. LC CIRCUIT



Current in an LC circuit oscilates according to:

$$I(t) = I_0 \sin(\omega_0 t),$$

where $\omega_0 = \frac{1}{\sqrt{LC}}$.

according to Faraday's law of induction. For a conductor which forms a closed loop, the emf sets up an induced current $I = |\varepsilon| / R$, where R is the resistance of the loop. To compute the induced current and its direction, we follow the procedure below:

1. For the closed loop of area A on a plane, define an area vector $\vec{\mathbf{A}}$ and let it point in the direction of your thumb, for the convenience of applying the right-hand rule later. Compute the magnetic flux through the loop using

$$\Phi_B = \begin{cases} \vec{\mathbf{B}} \cdot \vec{\mathbf{A}} & (\vec{\mathbf{B}} \text{ is uniform}) \\ \iint \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} & (\vec{\mathbf{B}} \text{ is non-uniform}) \end{cases}$$

Determine the sign of Φ_B.

2. Evaluate the rate of change of magnetic flux $d\Phi_B / dt$. Keep in mind that the change could be caused by

(i) changing the magnetic field $dB / dt \neq 0$,
(ii) changing the loop area if the conductor is moving ($dA / dt \neq 0$), or
(iii) changing the orientation of the loop with respect to the magnetic field ($d\theta / dt \neq 0$).

Determine the sign of $d\Phi_B / dt$.

3. The sign of the induced emf is the opposite of that of $d\Phi_B / dt$. The direction of the induced current can be found by using Lenz's law discussed in Section 10.1.2.

10.9 Solved Problems

10.9.1 Rectangular Loop Near a Wire

An infinite straight wire carries a current I is placed to the left of a rectangular loop of wire with width w and length l, as shown in the Figure 10.9.1.

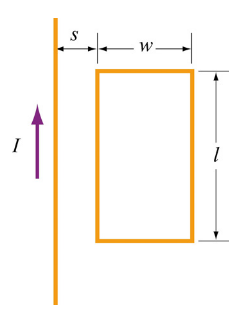


Figure 10.9.1 Rectangular loop near a wire

(a) Determine the magnetic flux through the rectangular loop due to the current *I*.

(b) Suppose that the current is a function of time with $I(t) = a + bt$, where a and b are positive constants. What is the induced emf in the loop and the direction of the induced current?

Solutions:

(a) Using Ampere's law:

$$\oint \vec{\mathbf{B}} \cdot d\,\vec{\mathbf{s}} = \mu_0 I_{\text{enc}} \qquad (10.9.1)$$

the magnetic field due to a current-carrying wire at a distance *r* away is

$$B = \frac{\mu_0 I}{2\pi r} \qquad (10.9.2)$$

The total magnetic flux Φ_B through the loop can be obtained by summing over contributions from all differential area elements $dA = l\,dr$:

$$\Phi_B = \int d\Phi_B = \int \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = \frac{\mu_0 I l}{2\pi} \int_s^{s+w} \frac{dr}{r} = \frac{\mu_0 I l}{2\pi} \ln\left(\frac{s+w}{s}\right) \qquad (10.9.3)$$

Note that we have chosen the area vector to point *into* the page, so that $\Phi_B > 0$.

(b) According to Faraday's law, the induced emf is

$$\varepsilon = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\left[\frac{\mu_0 I l}{2\pi} \ln\left(\frac{s+w}{s}\right)\right] = -\frac{\mu_0 l}{2\pi} \ln\left(\frac{s+w}{s}\right) \cdot \frac{dI}{dt} = -\frac{\mu_0 b l}{2\pi} \ln\left(\frac{s+w}{s}\right) \qquad (10.9.4)$$

where we have used $dI/dt = b$.

The straight wire carrying a current *I* produces a magnetic flux into the page through the rectangular loop. By Lenz's law, the induced current in the loop must be flowing *counterclockwise* in order to produce a magnetic field out of the page to counteract the increase in inward flux.

10.9.2 Loop Changing Area

A square loop with length *l* on each side is placed in a uniform magnetic field pointing into the page. During a time interval Δt, the loop is pulled from its two edges and turned into a rhombus, as shown in the Figure 10.9.2. Assuming that the total resistance of the loop is R, find the average induced current in the loop and its direction.

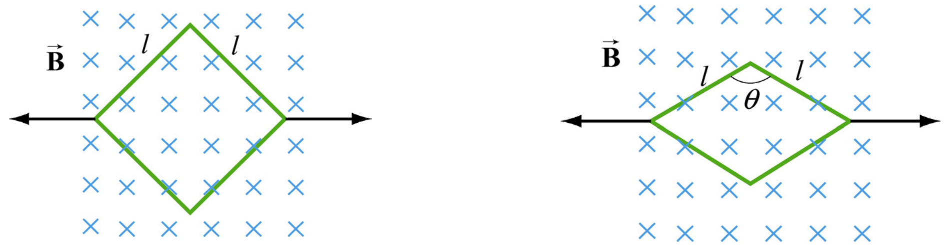


Figure 10.9.2 Conducting loop changing area

Solution:

Using Faraday's law, we have

$$\varepsilon = -\frac{\Delta\Phi_B}{\Delta t} = -B\left(\frac{\Delta A}{\Delta t}\right) \tag{10.9.5}$$

Since the initial and the final areas of the loop are $A_i = l^2$ and $A_f = l^2 \sin\theta$, respectively (recall that the area of a parallelogram defined by two vectors $\vec{l}_1$ and $\vec{l}_2$ is $A = |\vec{l}_1 \times \vec{l}_2| = l_1 l_2 \sin\theta$), the average rate of change of area is

$$\frac{\Delta A}{\Delta t} = \frac{A_f - A_i}{\Delta t} = -\frac{l^2(1-\sin\theta)}{\Delta t} < 0 \tag{10.9.6}$$

which gives

$$\varepsilon = \frac{Bl^2(1-\sin\theta)}{\Delta t} > 0 \tag{10.9.7}$$

Thus, the average induced current is

$$I = \frac{\varepsilon}{R} = \frac{Bl^2(1-\sin\theta)}{\Delta t R} \tag{10.9.8}$$

Since $(\Delta A/\Delta t) < 0$, the magnetic flux into the page decreases. Hence, the current flows in the clockwise direction to compensate the loss of flux.

10.9.3 Sliding Rod

A conducting rod of length l is free to slide on two parallel conducting bars as in Figure 10.9.3.

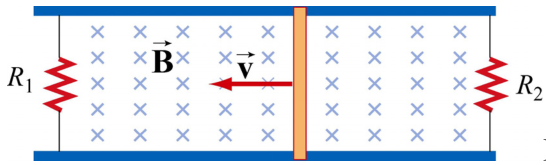


Figure 10.9.3 Sliding rod

In addition, two resistors R_1 and R_2 are connected across the ends of the bars. There is a uniform magnetic field pointing into the page. Suppose an external agent pulls the bar to the left at a constant speed v. Evaluate the following quantities:

(a) The currents through both resistors;

(b) The total power delivered to the resistors;

(c) The applied force needed for the rod to maintain a constant velocity.

Solutions:

(a) The emf induced between the ends of the moving rod is

$$\varepsilon = -\frac{d\Phi_B}{dt} = -Blv \tag{10.9.9}$$

The currents through the resistors are

$$I_1 = \frac{|\varepsilon|}{R_1}, \quad I_2 = \frac{|\varepsilon|}{R_2} \tag{10.9.10}$$

Since the flux into the page for the left loop is decreasing, I_1 flows clockwise to produce a magnetic field pointing into the page. On the other hand, the flux into the page for the right loop is increasing. To compensate the change, according to Lenz's law, I_2 must flow counterclockwise to produce a magnetic field pointing out of the page.

(b) The total power dissipated in the two resistors is

$$P_R = I_1|\varepsilon| + I_2|\varepsilon| = (I_1 + I_2)|\varepsilon| = \varepsilon^2\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = B^2l^2v^2\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{10.9.11}$$

(c) The total current flowing through the rod is $I = I_1 + I_2$. Thus, the magnetic force acting on the rod is

$$F_B = IlB = |\varepsilon|lB\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = B^2l^2v\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{10.9.12}$$

and the direction is to the right. Thus, an external agent must apply an equal but opposite force $\vec{\mathbf{F}}_{\text{ext}} = -\vec{\mathbf{F}}_B$ to the left in order to maintain a constant speed.

Alternatively, we note that since the power dissipated in the resistors must be equal to P_{ext}, the mechanical power supplied by the external agent. The same result is obtained since

$$P_{\text{ext}} = \vec{\mathbf{F}}_{\text{ext}} \cdot \vec{\mathbf{v}} = F_{\text{ext}} v \tag{10.9.13}$$

10.9.4 Moving Bar

A conducting rod of length l moves with a constant velocity $\vec{\mathbf{v}}$ perpendicular to an infinitely long, straight wire carrying a current I, as shown in the Figure 10.9.4. What is the emf generated between the ends of the rod?

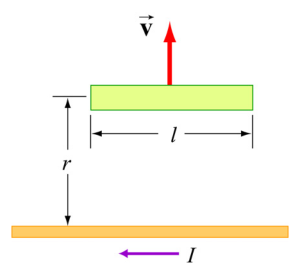


Figure 10.9.4 A bar moving away from a current-carrying wire

Solution:

From Faraday's law, the motional emf is

$$|\varepsilon| = Blv \tag{10.9.14}$$

where v is the speed of the rod. However, the magnetic field due to the straight current-carrying wire at a distance r away is, using Ampere's law:

$$B = \frac{\mu_0 I}{2\pi r} \tag{10.9.15}$$

Thus, the emf between the ends of the rod is given by

$$|\varepsilon| = \left(\frac{\mu_0 I}{2\pi r}\right) lv \tag{10.9.16}$$